Aufgabe 1:

$f : \mathbb{R} \to \mathbb{R}$ mit $f(x) = e^x$ für $x \in [-\pi, \pi]$

a)
$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x \cdot e^{-ikx}\, dx$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-ik)x}\, dx$$
$$= \frac{1}{2\pi} \left[\frac{e^{(1-ik)x}}{1-ik} \right]_{-\pi}^{\pi}$$
$$= \frac{1}{2\pi} \underbrace{\frac{1}{1-ik}}\left(e^{(1-ik)\pi} - e^{-(1-ik)\pi} \right)$$
$$= \frac{1+ik}{(1-ik)(1+ik)} = \frac{1+ik}{1-\underbrace{i^2}_{-1} k^2} = \frac{1+ik}{1+k^2}$$
$$= \frac{1}{2\pi} \frac{1+ik}{1+k^2} \left(e^{\pi} \cdot \underbrace{e^{-ik\pi}}_{(e^{-i\pi})^k = (-1)^k} - e^{-\pi} \underbrace{e^{ik\pi}}_{(e^{i\pi})^k = (-1)^k} \right)$$
$$= \frac{1}{2\pi} \frac{1+ik}{1+k^2} \left(e^{\pi}(-1)^k - e^{-\pi}(-1)^k \right)$$
$$= (-1)^k \frac{e^{\pi} - e^{-\pi}}{2\pi} \frac{1+ik}{1+k^2} \quad \text{für } k \in \mathbb{Z}$$

Ergebnis:
$$c_k = (-1)^k \frac{e^{\pi} - e^{-\pi}}{2\pi} \frac{1+ik}{1+k^2} \quad \text{für } k \in \mathbb{Z}$$
$$f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{k=1}^{\infty} (-1)^k \frac{1+ik}{1+k^2} e^{-ikx} \quad \text{für } x \in \mathbb{R}$$

b)
$$a_0 = 2c_0 = 2 \cdot \frac{e^{\pi} - e^{-\pi}}{2\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi}$$
$$a_k = c_k + c_{-k}$$

$$a_k = c_k + c_{-k}$$

$$= (-1)^k \frac{e^{\pi} - e^{-\pi}}{\cancel{2}\pi} \cdot \frac{\overbrace{(\cancel{1}+\cancel{ik}) + (1-\cancel{ik})}^{\cancel{2}}}{1+k^2}$$

$$= (-1)^k \frac{e^{\pi} - e^{-\pi}}{\pi(1+k^2)}$$

$$b_k = i(c_k - c_{-k})$$

$$= i(-1)^k \frac{e^{\pi} - e^{-\pi}}{\cancel{2}\pi} \cdot \frac{\overbrace{(\cancel{1}+ik) - (\cancel{1}-ik)}^{\cancel{2}ik}}{1+k^2}$$

$$= \underbrace{i \cdot i}_{-1} (-1)^k \frac{e^{\pi} - e^{-\pi}}{\pi(1+k^2)}$$

$$= (-1)^{k+1} \frac{e^{\pi} - e^{-\pi}}{\pi(1+k^2)} \cdot k$$

Ergebnis:

$$a_0 = \frac{e^{\pi} - e^{-\pi}}{\pi}$$

$$a_k = (-1)^k \frac{e^{\pi} - e^{-\pi}}{\pi(1+k^2)}$$

$$b_k = (-1)^k \frac{e^{\pi} - e^{-\pi}}{\pi(1+k^2)} \cdot k$$

$$f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \frac{e^{\pi} - e^{-\pi}}{\pi} \sum_{k=1}^{\infty} (-1)^k \left(\frac{\cos(kx)}{1+k^2} - \frac{k \cdot \sin(kx)}{1+k^2} \right)$$

Aufgabe 2

a) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$f^{\wedge}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-iux} \, dx$$

$$1 \int^{\infty} e^{-x} e^{-iux} \, dx$$

$$= \frac{1}{2\pi} \int_0^\infty \underbrace{e^{-x} e^{-iux}}_{e^{-(1+iu)x}} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{-(1+iu)x}}{-(1+iu)} \right]_0^\infty$$

$$= -\frac{1}{2\pi(1+iu)} \Big(\underbrace{e^{-\infty}}_{0} - \underbrace{e^{0}}_{1} \Big)$$

$$= \frac{1}{2\pi(1+iu)} \quad \text{für } u \in \mathbb{R}.$$

b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-|x|}$

$$f^\wedge(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} e^{-iux} dx$$

$$= \frac{1}{2\pi} \Big(\int_{-\infty}^{0} \underbrace{e^{-\overbrace{|x|}^{-x}}}_{e^{x}} e^{-iux} dx + \int_0^\infty \underbrace{e^{-\overbrace{|x|}^{x}}}_{e^{-x}} e^{-iux} dx \Big)$$

$$= \frac{1}{2\pi} \Big(\int_{-\infty}^{0} \underbrace{e^{x} e^{-iux}}_{e^{(1-iu)x}} dx + \int_0^\infty \underbrace{e^{-x} e^{-iux}}_{e^{(-1-iu)x}} dx \Big)$$

$$= \frac{1}{2\pi} \Big(\left[\frac{e^{(1-iu)x}}{1-iu} \right]_{-\infty}^{0} + \left[\frac{e^{(-1-iu)x}}{-1-iu} \right]_0^\infty \Big)$$

$$= \frac{1}{2\pi} \Big(\frac{1}{1-iu} \big(\underbrace{e^{-\infty}}_{0} - \underbrace{e^{0}}_{1} \big) + \frac{1}{-1-iu} \big(\underbrace{e^{-\infty}}_{0} - \underbrace{e^{0}}_{1} \big) \Big)$$

$$= \frac{1}{2\pi} \Big(\underbrace{\frac{1}{1-iu} - \frac{1}{-1-iu}}_{} \Big)$$

$$\frac{-1-\cancel{iu}-(1-\cancel{iu})}{} = \frac{-2}{}$$

$$\frac{\underbrace{-1-\cancel{iu}-(1-\cancel{iu})}}{(1-iu)\cdot(-1)\cdot(1+iu)} = \frac{-2}{(-1)\,(1-\underbrace{i^2}_{-1}u^2)}$$

$$= \frac{1}{\cancel{2}\pi}\cdot\frac{\cancel{2}}{1+u^2}$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+u^2} \quad \text{für } u\in\mathbb{R}$$

c) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} xe^{-x}, & x\geq 0\\ 0, & x<0\end{cases}$

$$f^\wedge(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\,e^{-iux}\,dx$$

$$= \frac{1}{2\pi}\int_0^\infty x\cdot\underbrace{e^{-x}e^{-iux}}_{e^{-(1+iu)x}}\,dx$$

$$= \frac{1}{2\pi}\int_0^\infty \underbrace{x}_{u}\cdot\underbrace{e^{-(1+iu)x}}_{v'}\,dx$$

$$= \frac{1}{2\pi}\left(\left[\underbrace{x}_{u}\cdot\underbrace{\frac{e^{-(1+iu)x}}{-(1+iu)}}_{v}\right]_0^\infty - \int_0^\infty \underbrace{1}_{u'}\cdot\underbrace{\frac{e^{-(1+iu)x}}{-(1+iu)}}_{v}\,dx\right)$$

$$= \frac{1}{2\pi}\left(\underbrace{\frac{1}{-(1+iu)}\left(\underbrace{\infty\cdot e^{-\infty}}_{0}-0\right)}_{0} - \left[\frac{e^{-(1+iu)x}}{(1+iu)^2}\right]_0^\infty\right)$$

wegen $te^{-t}\to 0$ für $t\to\infty$

$$= -\frac{1}{2\pi}\,\frac{1}{(1+iu)^2}\left(\underbrace{e^{-\infty}}-\underbrace{e^0}\right)$$

$$= -\frac{1}{2\pi}\frac{1}{(1+iu)^2}\left(\underbrace{e^{-\infty}}_{0} - \underbrace{e^{0}}_{1}\right)$$

$$= \frac{1}{2\pi}\frac{1}{(1+iu)^2} \text{ für } u \in \mathbb{R}.$$

Lösung zu Aufgabe 3

a) $g: \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ mit $a > 0$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$f^\wedge(u) = \frac{1}{2\pi}\frac{1}{1+iu}, \quad u \in \mathbb{R}$$

Es gilt:

$$g(x) = f(a \cdot x)$$

$$g^\wedge(u) = \frac{1}{a} f^\wedge\left(\frac{u}{a}\right)$$

$$= \frac{1}{a} \cdot \frac{1}{2\pi}\frac{1}{1+i\frac{u}{a}}$$

$$= \frac{1}{2\pi}\frac{1}{a+iua} \quad \text{für } u \in \mathbb{R}$$

b) $g: \mathbb{R} \to \mathbb{R}$, $g(x) = e^{-|x-a|}$ mit $a \in \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-|x|}$

$$f^\wedge(u) = \frac{1}{\pi}\frac{1}{1+u^2}, \quad u \in \mathbb{R}$$

$$f^\wedge(u) = \frac{1}{\pi}\frac{1}{1+u^2}, \quad u\in\mathbb{R}$$

Es gilt

$$g(x) = f(x-a)$$

$$g^\wedge(x) = e^{-iua}\cdot f^\wedge(u)$$

$$= e^{-iua}\cdot\frac{1}{\pi}\frac{1}{1+u^2} \quad \text{für } u\in\mathbb{R}$$

c) $g:\mathbb{R}\to\mathbb{R}$, $g(x) = \begin{cases}(1+x)e^{-x}, & x\geq 0\\ 0, & x<0\end{cases}$

$$f:\mathbb{R}\to\mathbb{R},\ f(x) = \begin{cases}e^{-x}, & x\geq 0\\ 0, & x<0\end{cases}$$

$$f^\wedge(u) = \frac{1}{2\pi}\frac{1}{1+iu}, \quad u\in\mathbb{R}$$

$$h:\mathbb{R}\to\mathbb{R},\ h(x) = \begin{cases}xe^{-x}, & x\geq 0\\ 0, & x<0\end{cases}$$

$$h^\wedge(u) = \frac{1}{2\pi}\frac{1}{(1+iu)^2}, \quad u\in\mathbb{R}$$

Es gilt:

$$g(x) = f(x) + h(x)$$

$$g^\wedge(u) = f^\wedge(u) + h^\wedge(u)$$

$$= \frac{1}{2\pi}\frac{1}{1+iu} + \frac{1}{2\pi}\frac{1}{(1+iu)^2} \quad \text{für } u\in\mathbb{R}$$