Übungen zur Mathematik 2

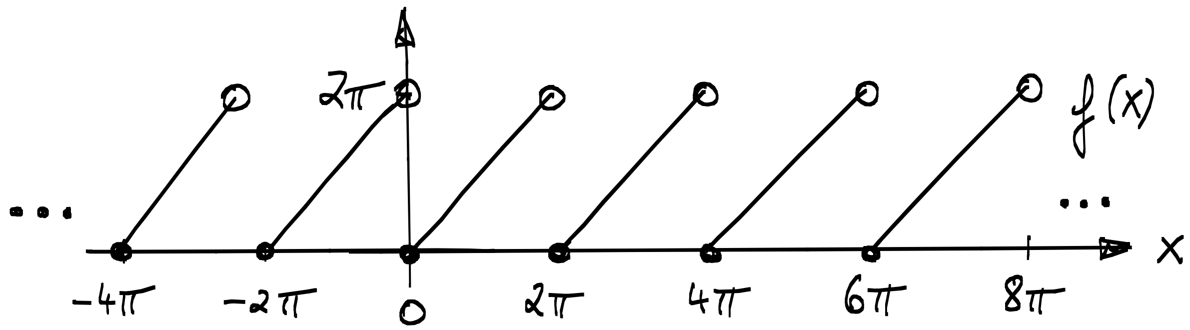
Lösungen Fourier-Reihen 1

Aufgabe 1

$f(x): \mathbb{R} \to \mathbb{R}$ 2π-periodisch

$f(x) = x$ für $x \in [0, 2\pi)$

a)



f ist weder gerade noch ungerade

b) $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(0\cdot x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_0^{2\pi} = \frac{1}{\pi}\left(2\pi^2 - 0\right) = \underline{\underline{2\pi}}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x}_{u}\cdot\underbrace{\cos(kx)}_{v'}\,dx \quad \text{mit } k \in \mathbb{N}$$

partielle Integration $\xrightarrow{k \neq 0}$ $$= \frac{1}{\pi}\left\{\left[\underbrace{x}_{u}\cdot\underbrace{\frac{1}{k}\sin(kx)}_{v}\right]_0^{2\pi} - \int_0^{2\pi} \underbrace{1}_{u'}\cdot\underbrace{\frac{1}{k}\sin(kx)}_{v}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left(2\pi\cdot\frac{1}{k}\underbrace{\sin(k\cdot 2\pi)}_{=0} - 0\right) - \left[\frac{-1}{k^2}\cos(kx)\right]_0^{2\pi}\right\}$$

$$= 0 + \frac{1}{\pi k^2}\left(\underbrace{\cos(k\cdot 2\pi)}_{=1} - \underbrace{\cos(0)}_{=1}\right) = \underline{\underline{0}}$$

Erinnerung partielle Integration:

$$\int_a^b u(x)\,v'(x)\,dx = \big[u(x)\,v(x)\big]_a^b - \int_a^b u'(x)\,v(x)\,dx$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x}_{u} \cdot \underbrace{\sin(kx)}_{v'}\,dx \quad \text{mit } k \in \mathbb{N}$$

partielle Integration $\xrightarrow{k \neq 0}$
$$= \frac{1}{\pi}\Big[\underbrace{x}_{u} \cdot \underbrace{\frac{-1}{k}\cos(kx)}_{v}\Big]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi} \underbrace{1}_{u'} \cdot \underbrace{\frac{-1}{k}\cos(kx)}_{v}\,dx$$

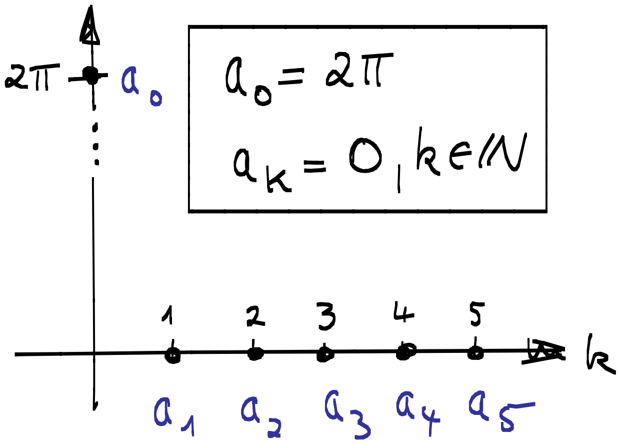
$$= \frac{1}{\cancel{\pi}}\Big(2\cancel{\pi} \cdot \frac{-1}{k}\underbrace{\cos(k \cdot 2\pi)}_{=1} - 0\Big) + \frac{1}{\pi}\Big[\frac{1}{k^2}\sin(kx)\Big]_0^{2\pi}$$

$$= -\frac{2}{k} + \frac{1}{\pi}\Big(\frac{1}{k^2}\underbrace{\sin(k \cdot 2\pi)}_{=0} - 0\Big)$$

$$= \underline{\underline{-\frac{2}{k}}}$$

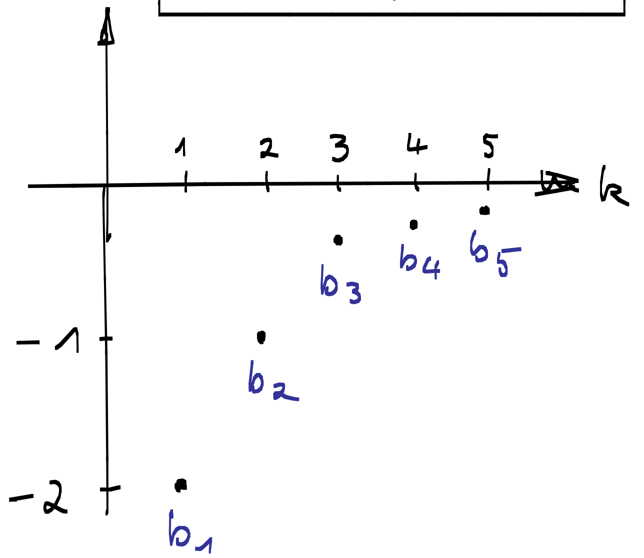
Fourier-Koeffizienten grafisch:

$$\boxed{\begin{aligned} a_0 &= 2\pi \\ a_k &= 0,\ k \in \mathbb{N} \end{aligned}}$$



Koeffizienten Null

$$\boxed{b_k = -\frac{2}{k},\ k \in \mathbb{N}}$$



Koeffizienten konvergieren gegen Null: $b_k \to 0\ (k \to \infty)$

c) Aus den Fourier-Koeffizienten läßt sich die Funktion f rekonstruieren.

Fourier-Reihe:

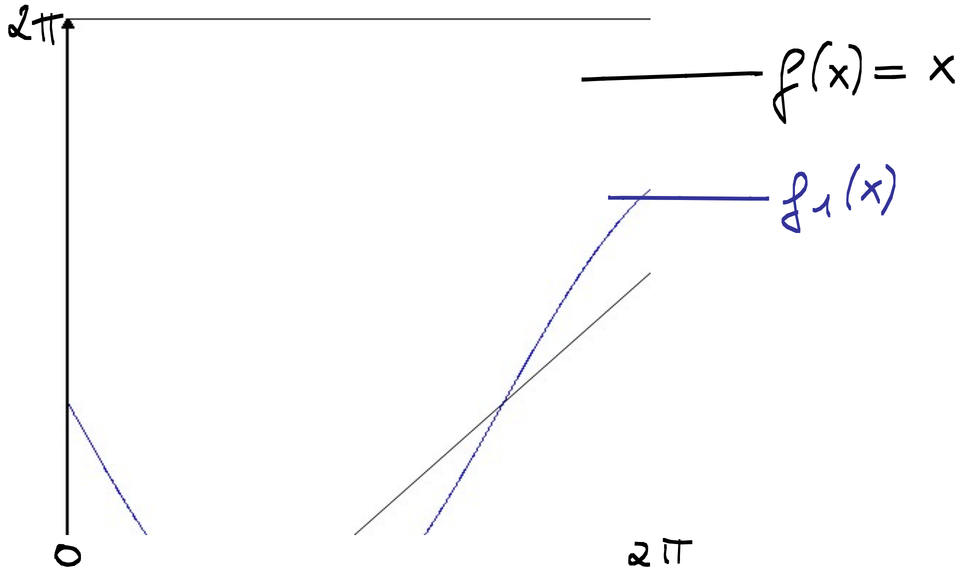
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(\underbrace{a_k}_{=0} \cos(kx) + b_k \sin(kx) \right)$$

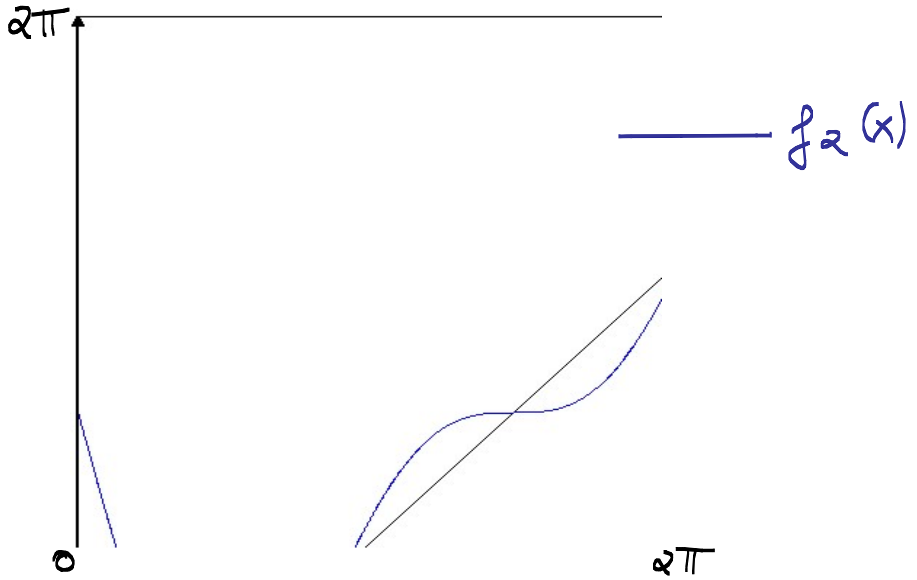
$$= \pi - \sum_{k=1}^{\infty} \frac{2}{k} \sin(kx)$$

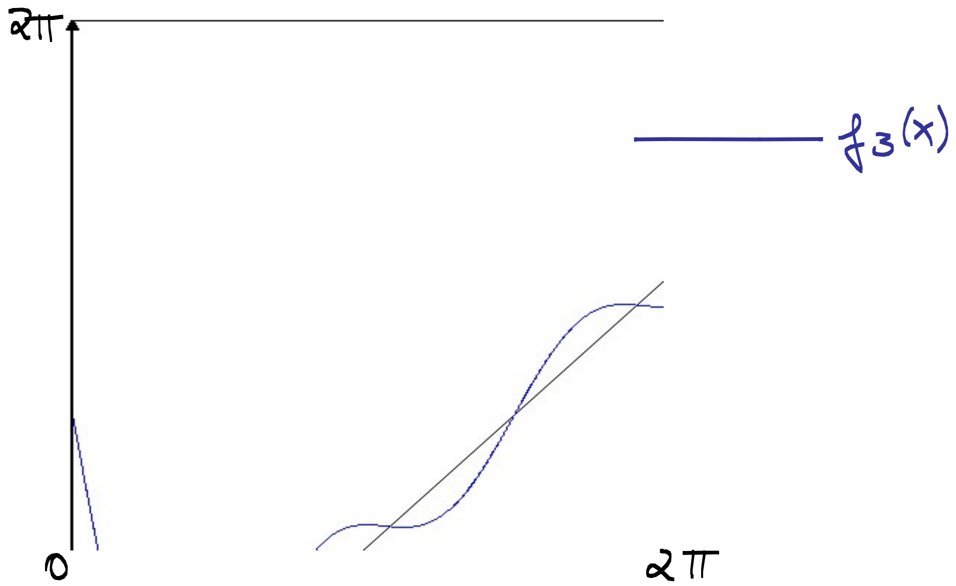
c) 1. Näherung : $f_1(x) = \pi - 2\sin(x)$

2. Näherung : $f_2(x) = \pi - 2\sin(x) - \sin(2x)$

3. Näherung : $f_3(x) = \pi - 2\sin(x) - \sin(2x) - \frac{2}{3}\sin(3x)$



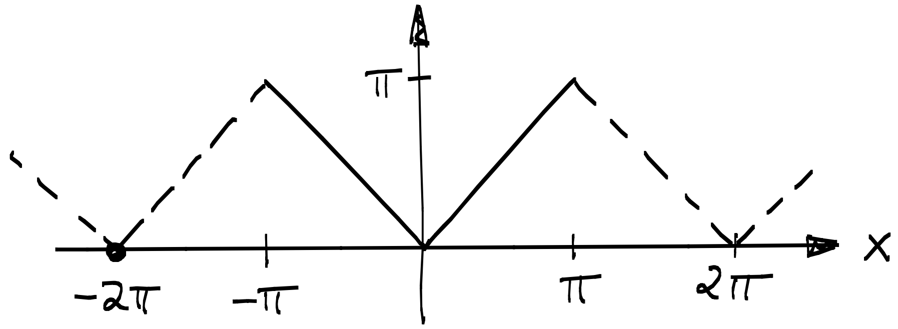




Aufgabe 2

$f: \mathbb{R} \to \mathbb{R}$ 2π- periodisch

$f(x) = |x|$ für $x \in [-\pi, \pi]$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx = 2 \cdot \frac{1}{\pi} \int_0^{\pi} x \, dx = \cancel{2} \cdot \frac{1}{\pi} \left[\frac{1}{\cancel{2}} x^2 \right]_0^{\pi}$$

$$= \frac{1}{\pi} (\pi^2 - 0) = \underline{\underline{\pi}}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{|x| \cdot \cos(kx)}_{\text{gerade}} \, dx \quad \text{mit } k \in \mathbb{N}$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{x}_{u} \, \underbrace{\cos(kx)}_{v'} \, dx \quad \text{partielle Integration}$$

$$\overset{k \neq 0}{=} \frac{2}{\pi} \left[x \cdot \frac{1}{k} \sin(kx) \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \frac{1}{k} \sin(kx) \, dx$$

$$= \frac{2}{\pi} \frac{1}{k} \Big(\underbrace{\sin(k\pi)}_{=0} - \underbrace{\sin(0)}_{=0} \Big) + \frac{2}{\pi} \cdot \frac{1}{k^2} \left[\cos(kx) \right]_0^{\pi}$$

$$= 0 + \frac{2}{\pi}\frac{1}{k^2}\big(\underbrace{\cos(k\cdot\pi)}_{(-1)^k} - 1\big)$$

$$= \frac{2}{\pi}\cdot\frac{1}{k^2}\cdot\begin{cases} 0, & k = 2,4,6,\dots \\ -2, & k = 1,3,5,\dots \end{cases}$$

$$= \begin{cases} 0, & k = 2,4,6,\dots \\ -\frac{4}{\pi k^2}, & k = 1,3,5,\dots \end{cases}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|x|\cdot\sin(kx)}_{\text{ungerade}}\,dx = \underline{\underline{0}}$$

Fourier-Reihe

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\big(a_k\cos(kx) + \underbrace{b_k}_{=0}\sin(kx)\big)$$

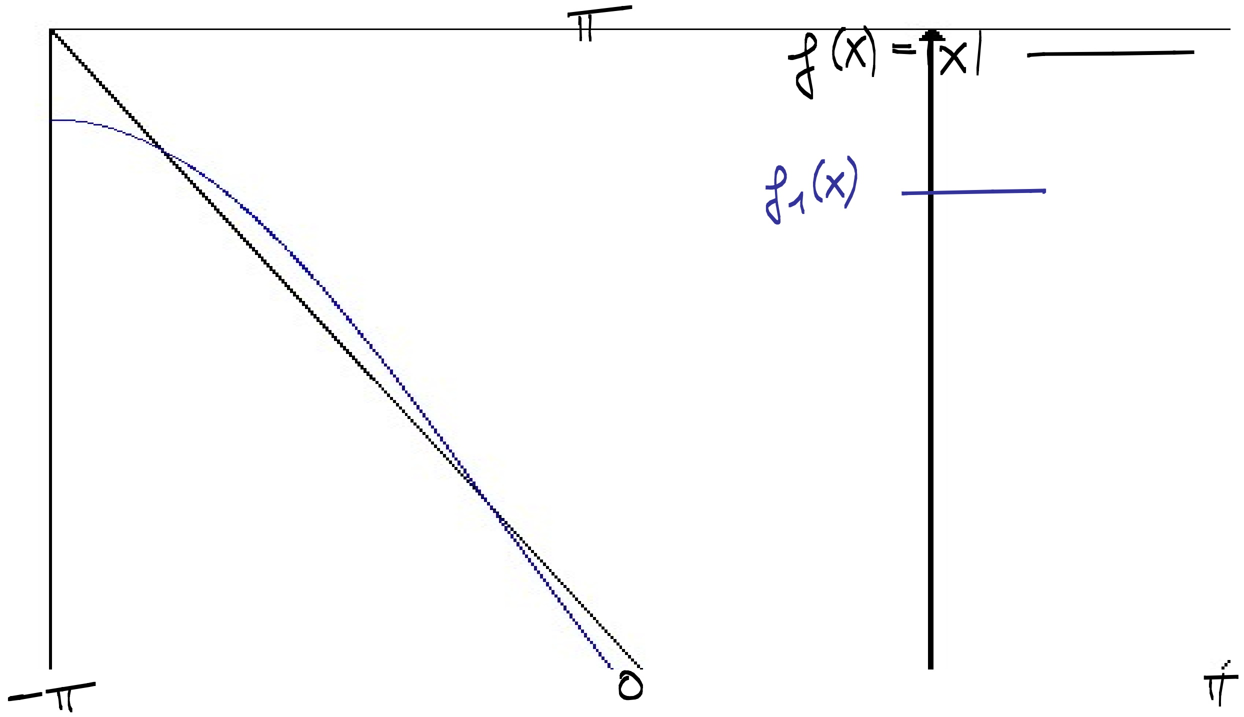
$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \dots\right)$$

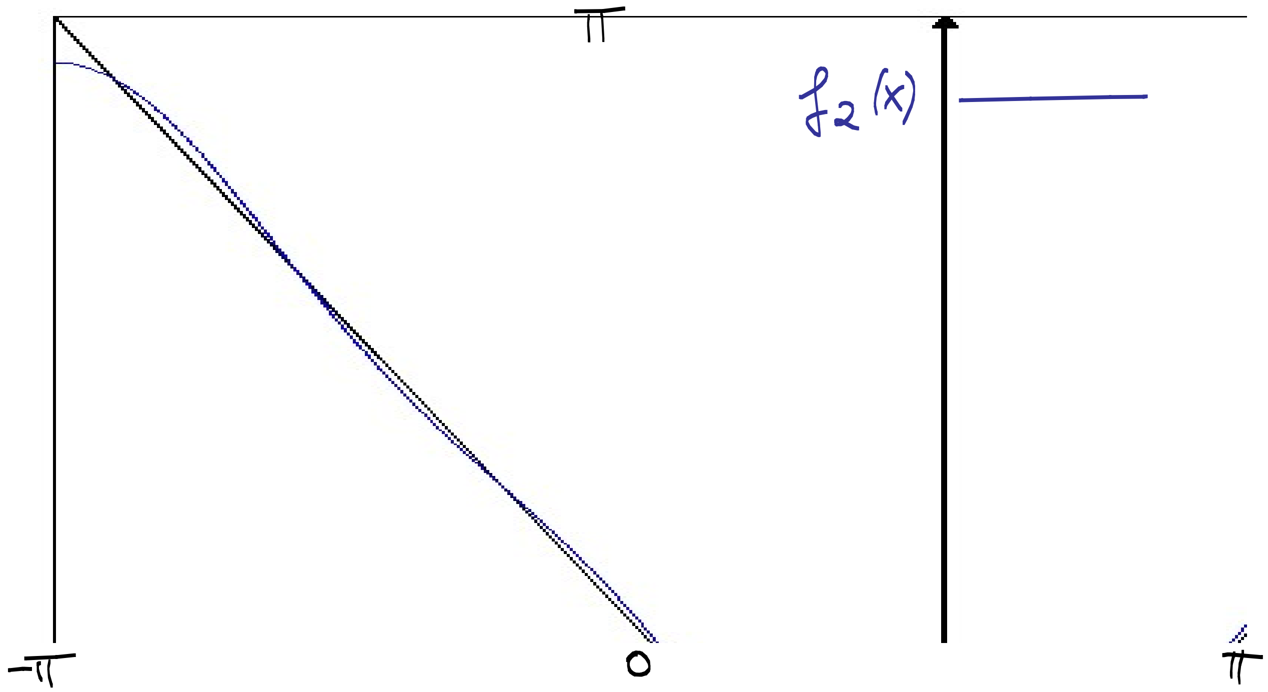
$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)x)$$

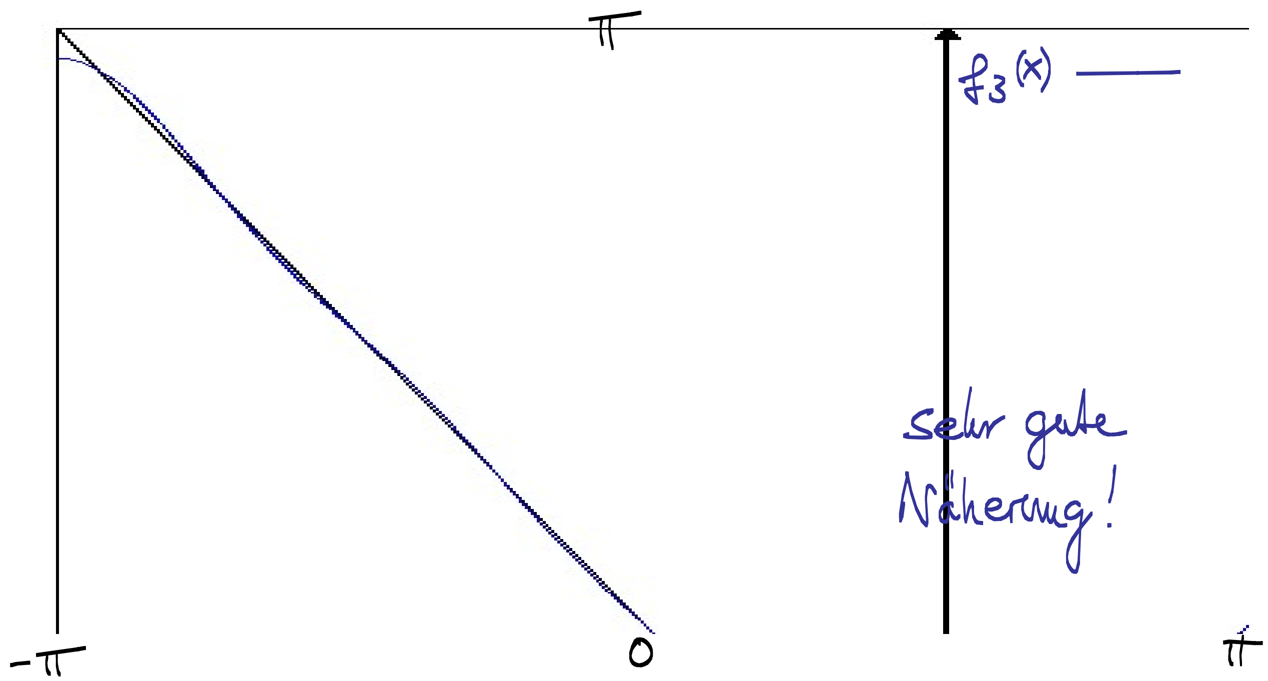
1. Näherung: $f_1(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x)$

2. " : $f_2(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x)$

3. " : $f_3(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - \frac{4}{25\pi}\cos(5x)$


π
$f(x) = |x|$
$f_1(x)$
$-\pi$
0
π


π
$f_2(x)$
$-\pi$
0
π


π
$f_3(x)$
sehr gute
Näherung!
$-\pi$
0
π