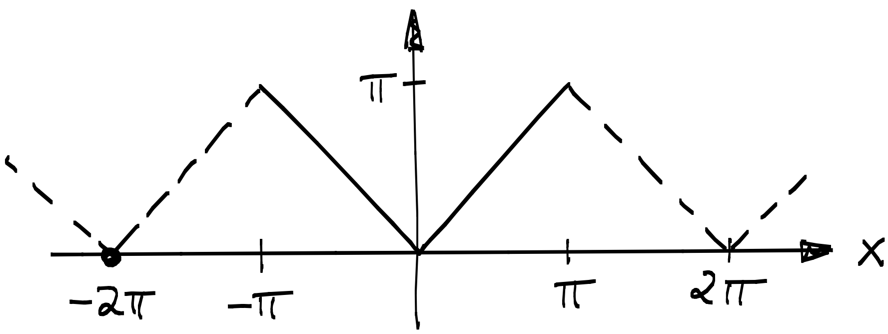
Übungen zur Mathematik

Lösungen Blatt 10

Aufgabe 1

$f: \mathbb{R} \to \mathbb{R}$ 2π- periodisch

$f(x) = |x|$ für $x \in [-\pi, \pi]$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \, dx = 2 \cdot \frac{1}{\pi} \int_0^{\pi} x \, dx = \cancel{2} \cdot \frac{1}{\pi} \left[\frac{1}{\cancel{2}} x^2 \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left(\pi^2 - 0 \right) = \underline{\underline{\pi}}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{|x| \cdot \cos(kx)}_{\text{gerade}} \, dx \quad \text{mit } k \in \mathbb{N}$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{x}_{u} \underbrace{\cos(kx)}_{v'} \, dx \quad \text{partielle Integration}$$

$$= \frac{2}{\pi} \left[x \cdot \frac{1}{k} \sin(kx) \right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \frac{1}{k} \sin(kx) \, dx$$

$$= \frac{2}{\pi} \frac{1}{k} \Big(\underbrace{\sin(k\pi)}_{=0} - \underbrace{\sin(0)}_{=0} \Big) + \frac{2}{\pi} \cdot \frac{1}{k^2} \left[\cos(kx) \right]_0^{\pi}$$

$$= 0 + \frac{2}{\pi}\frac{1}{k^2}\left(\underbrace{\cos(k\cdot\pi)}_{(-1)^k} - 1\right)$$

$$= \frac{2}{\pi}\cdot\frac{1}{k^2}\cdot\begin{cases} 0, & k = 2,4,6,\dots \\ -2, & k = 1,3,5,\dots \end{cases}$$

$$= \begin{cases} 0, & k = 2,4,6,\dots \\ -\frac{4}{\pi k^2}, & k = 1,3,5,\dots \end{cases}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{|x|\cdot\sin(kx)}_{\text{ungerade}}\,dx = \underline{\underline{0}}$$

Fourier-Reihe

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(kx) + \underbrace{b_k}_{=0}\sin(kx)\right)$$

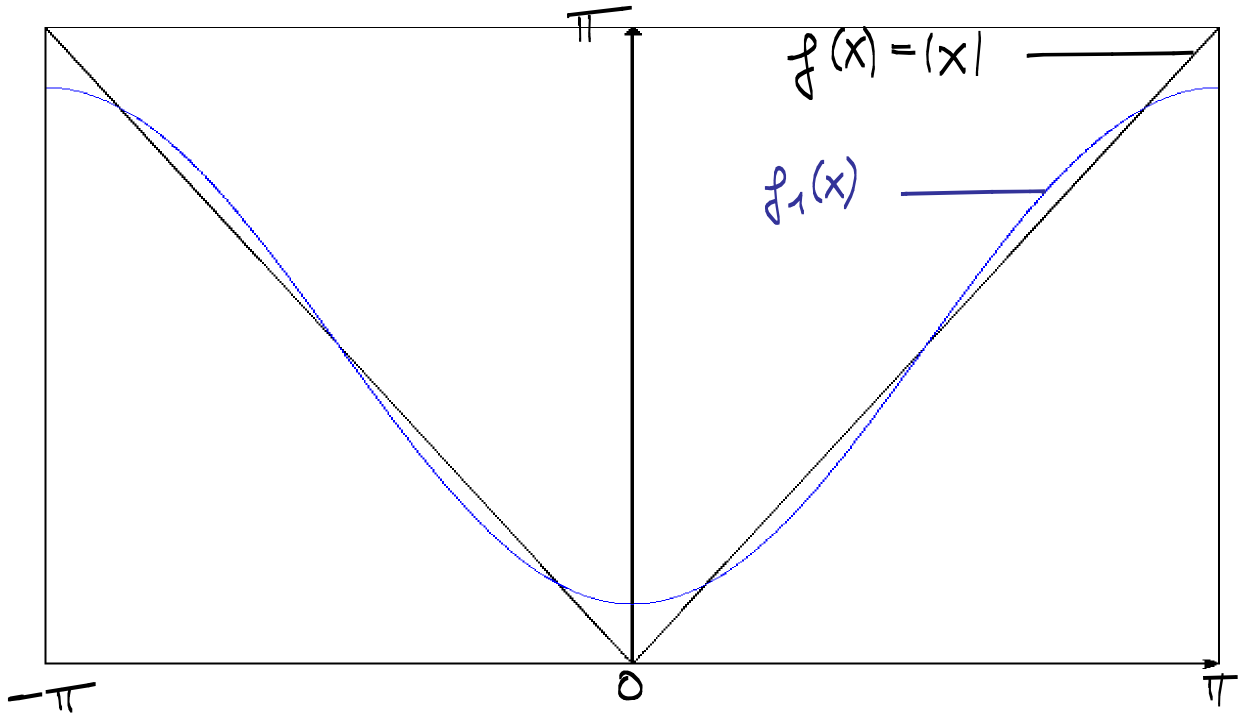
$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \dots\right)$$

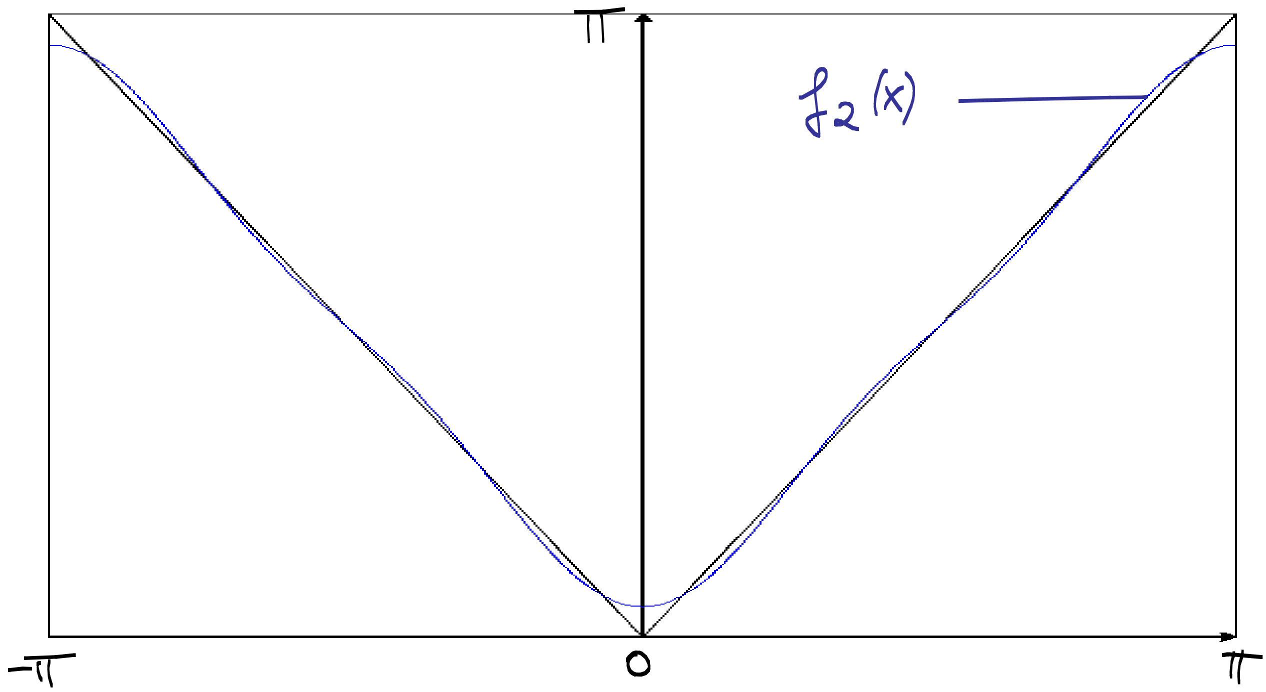
$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((2k-1)x)$$

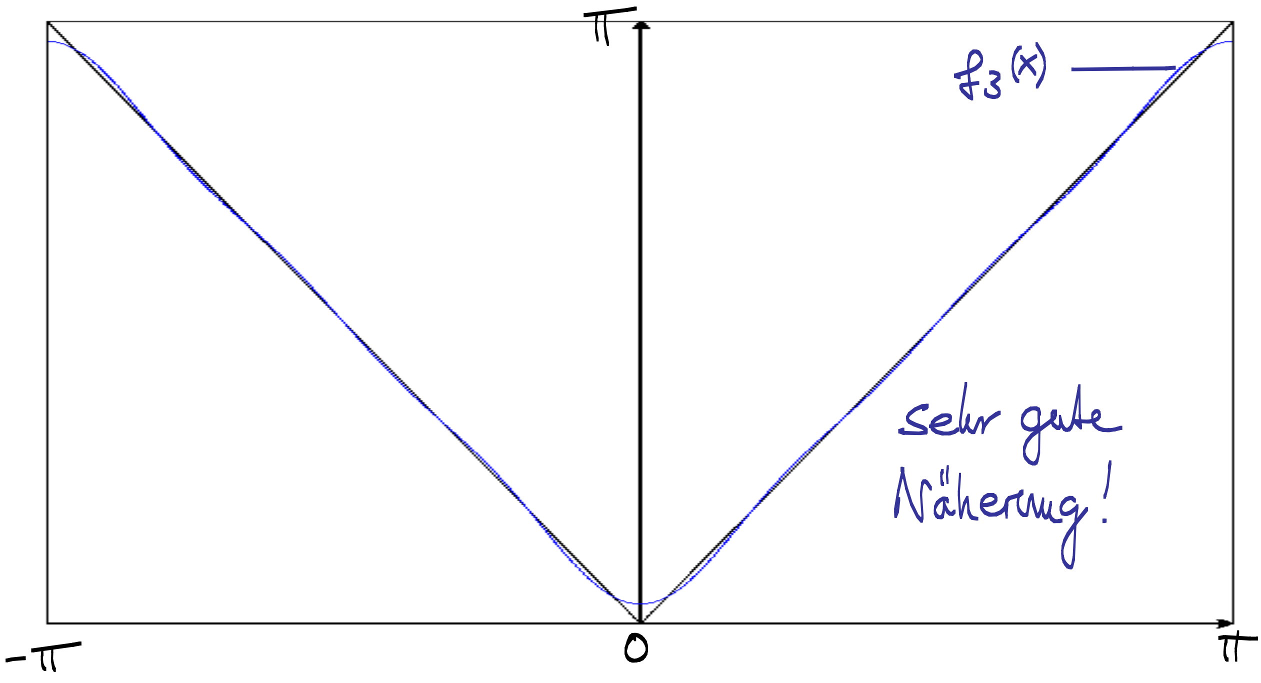
1. Näherung: $f_1(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x)$

2. " : $f_2(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x)$

3. " : $f_3(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos(x) - \frac{4}{9\pi}\cos(3x) - \frac{4}{25\pi}\cos(5x)$

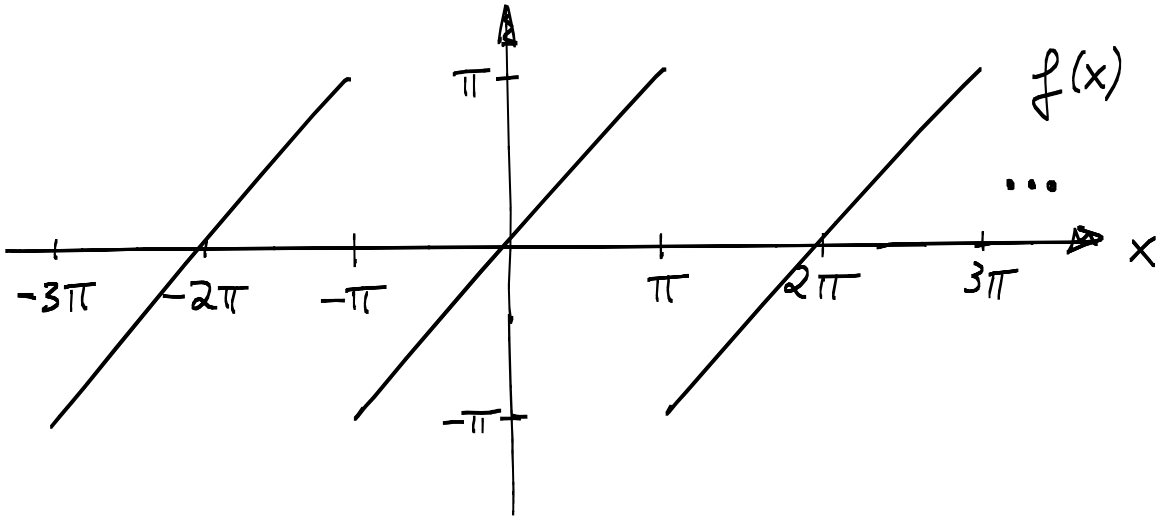

π
f(x) = |x|
f1(x)
−π
0
π


π
f2(x)
−π
0
π


π
f3(x)
sehr gute
Näherung!
−π
0
π

Aufgabe 2

Sägezahnfunktion



$f(x) = x$ für $-\pi < x < \pi$

$a_n = 0$ für alle $n \in \mathbb{N}_0$, da f ungerade

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \sin(kx)\,dx \quad \text{mit } k \in \mathbb{N}$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{x}_{u} \cdot \underbrace{\sin(kx)}_{v'}\,dx$$

$$= \frac{2}{\pi}\left[x \cdot \frac{-1}{k}\cos(kx)\right]_0^{\pi} - \frac{2}{\pi} \cdot \frac{1}{k} \int_0^{\pi} (-\cos(kx))\,dx$$

$$= -\frac{2}{k\cancel{\pi}}\left(\cancel{\pi}\underbrace{\cos(k\pi)}_{(-1)^k} - 0\right) + \frac{2}{k\pi} \cdot \frac{1}{k} \underbrace{\left[\sin(kx)\right]_0^{\pi}}_{=0}$$

da $\sin(k\pi) = 0$

$$= 2 \cdot \frac{(-1)^{k+1}}{k}$$

Fourier-Reihe

$$f(x) = \underbrace{\frac{a_0}{2}}_{=0} + \sum_{k=1}^{\infty} \left(\underbrace{a_k}_{=0} \cos(kx) + b_k \sin(kx) \right)$$

$$= 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)$$

$$= 2 \left(\sin(x) - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) \mp \dots \right)$$

Aufgabe 3

$f(x) = x^2$ für $x \in [-\pi, \pi]$

a) Da f gerade ist, sind alle $b_k = 0$ $(k \in \mathbb{N})$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x^2}_{\text{gerade}} dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{1}{3} x^3 \right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}} \frac{1}{3} \pi^{\cancel{3}\,2} = \underline{\underline{\frac{2}{3} \pi^2}}$$

Es sei $k \in \mathbb{N}$.

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x^2 \cos(kx)}_{\text{gerade}} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{x^2}_{u} \underbrace{\cos(kx)}_{v'} dx$$

$$\overset{k \neq 0}{=} \frac{2}{\pi} \left(\underbrace{\left[\frac{1}{k} x^2 \sin(kx) \right]_0^{\pi}}_{= 0} - \frac{2}{k} \int_0^{\pi} \underbrace{x}_{u} \underbrace{\sin(kx)}_{v'} dx \right)$$

$$= -\frac{4}{\pi k} \left(\left[\frac{-1}{k} x \cos(kx) \right]_0^{\pi} - \int_0^{\pi} \frac{-1}{k} \cos(kx) dx \right)$$

$$= -\frac{4}{\pi k}\left(\frac{-1}{k}\pi \underbrace{\cos(k\pi)}_{(-1)^k} - 0 + \frac{1}{k}\underbrace{\int_0^\pi \cos(kx)\,dx}_{\left[\frac{1}{k}\sin(kx)\right]_0^\pi}\right)$$

$$= -\frac{4}{\pi k}\left(\frac{(-1)^{k+1}}{k}\pi + \frac{1}{k^2}\left(\underbrace{\sin(k\pi)}_{=0} - 0\right)\right)$$

$$= -\frac{4}{\cancel{\pi} k}\frac{(-1)^{k+1}}{k}\cancel{\pi} = (-1)^{k+2}\frac{4}{k^2}$$

$$= (-1)^k \frac{4}{k^2}$$

Damit erhalten wir die Fourier-Reihe

$$f(x) = \underbrace{\frac{\pi^2}{3} + 4 \cdot \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(kx)}_{\text{Cosinus-Reihe, da } f \text{ gerade}} \qquad (*)$$