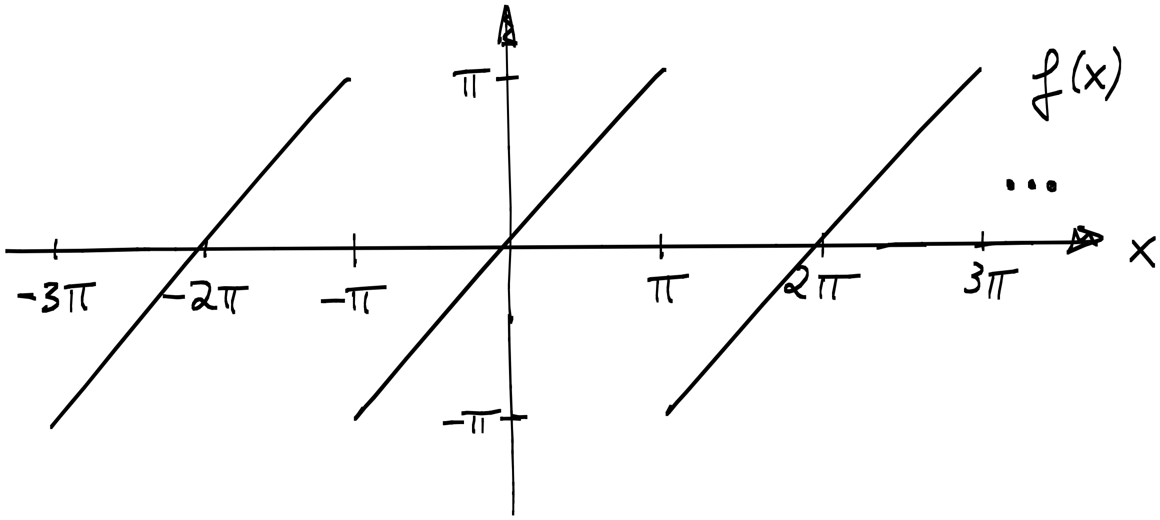
Übungen zur Mathematik 2

Lösungen Blatt 5

Aufgabe 1

Sägezahnfunktion



$$f(x) = x \quad \text{für} \quad -\pi < x < \pi$$

$a_n = 0$ für alle $n \in \mathbb{N}_0$, da f ungerade

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \sin(kx)\,dx \quad \text{mit } k \in \mathbb{N}$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \underbrace{x}_{u} \cdot \underbrace{\sin(kx)}_{v'}\,dx$$

$$= \frac{2}{\pi}\left[x \cdot \frac{-1}{k}\cos(kx)\right]_0^{\pi} - \frac{2}{\pi}\int_0^{2} 1 \cdot \frac{-1}{k}\cos(kx)\,dx$$

$$= -\frac{2}{k\cancel{\pi}}\Big(\cancel{\pi}\underbrace{\cos(k\pi)}_{(-1)^k} - 0\Big) + \frac{2}{k\pi}\cdot\frac{1}{k}\underbrace{\left[\sin(kx)\right]_0^{\pi}}_{\substack{=0 \\ \text{da } \sin(k\pi)=0}}$$

$$= 2 \cdot \frac{(-1)^{k+1}}{k}$$

Fourier-Reihe

$$f(x) = \underbrace{\frac{a_0}{2}}_{=0} + \sum_{k=1}^{\infty}\big(\underbrace{a_k}_{=0}\cos(kx) + b_k\sin(kx)\big)$$

$$= 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(kx)$$

$$= 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) \mp \dots\right)$$

Aufgabe 2

$f(x) = x^2$ für $x \in [-\pi, \pi]$

a) Da f gerade ist, sind alle $b_k = 0$ $(k \in \mathbb{N})$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x^2}_{\text{gerade}} dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{1}{3} x^3 \right]_0^{\pi}$$

$$= \frac{2}{\cancel{\pi}} \frac{1}{3} \pi^{\cancel{3}2} = \underline{\underline{\frac{2}{3} \pi^2}}$$

Es sei $k \in \mathbb{N}$.

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x^2 \cos(kx)}_{\text{gerade}} dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \underbrace{x^2}_{u} \underbrace{\cos(kx)}_{v'} dx$$

$$\overset{k \neq 0}{=} \frac{2}{\pi} \Bigg(\underbrace{\left[\frac{1}{k} x^2 \sin(kx) \right]_0^{\pi}}_{= 0} - \frac{2}{k} \int_0^{\pi} \underbrace{x}_{u} \underbrace{\sin(kx)}_{v'} dx \Bigg)$$

$$= -\frac{4}{\pi k} \left(\left[\frac{-1}{k} x \cos(kx) \right]_0^{\pi} - \int_0^{\pi} \frac{-1}{k} \cos(kx) dx \right)$$

$$= -\frac{4}{\pi k}\left(\frac{-1}{k}\pi \underbrace{\cos(k\pi)}_{(-1)^k} - 0 + \frac{1}{k}\underbrace{\int_0^\pi \cos(kx)\,dx}_{\left[\frac{1}{k}\sin(kx)\right]_0^\pi}\right)$$

$$= -\frac{4}{\pi k}\left(\frac{(-1)^{k+1}}{k}\pi + \frac{1}{k^2}\left(\underbrace{\sin(k\pi)}_{=0} - 0\right)\right)$$

$$= -\frac{4}{\cancel{\pi} k}\,\frac{(-1)^{k+1}}{k}\cancel{\pi} = (-1)^{k+2}\,\frac{4}{k^2}$$

$$= \underline{\underline{(-1)^k\,\frac{4}{k^2}}}$$

Damit erhalten wir die Fourier-Reihe

$$f(x) = \underbrace{\frac{\pi^2}{3} + 4\cdot\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kx)}_{\text{Cosinus-Reihe, da } f \text{ gerade}} \qquad (*)$$