Übungen zur Mathematik 1
Lösungen Blatt 12

Aufgabe 1

$$
\begin{aligned}
(f\circ g)(x) &= f(g(x)) = g(x)^2 + g(x) - 1\\
&= (3x-1)^2 + (3x-1) - 1\\
&= 9x^2 - 6x + 1 + 3x - 1 - 1\\
&= 9x^2 - 3x - 1
\end{aligned}
$$

$$
\begin{aligned}
(g\circ f)(x) &= g(f(x)) = 3f(x) - 1\\
&= 3(x^2+x-1) - 1\\
&= 3x^2 + 3x - 4
\end{aligned}
$$

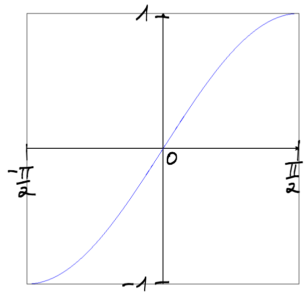
$$
\begin{aligned}
(f\circ f)(x) &= f(f(x)) = f(x)^2 + f(x) - 1\\
&= (x^2+x-1)^2 + (x^2+x-1) - 1\\
&= x^4 + 2x^3 - \cancel{2x^2} + \cancel{x^2} - 2x + 1 + \cancel{x^2} + x - 1 - 1\\
&= x^4 + 2x^3 - x - 1
\end{aligned}
$$

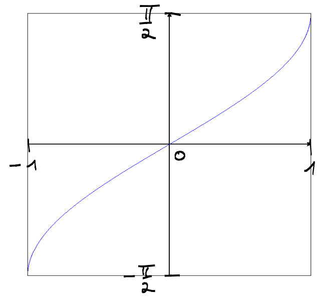
$$
\begin{aligned}
(g\circ g)(x) &= g(g(x)) = 3g(x) - 1\\
&= 3(3x-1) - 1\\
&= 9x - 4
\end{aligned}
$$

Aufgabe 2

a) $f_1 : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ $\quad$ $f_1^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

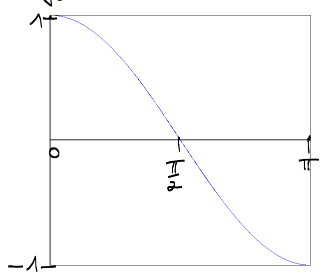
$f_1(x) = \sin(x)$ $\quad$ $f_1^{-1}(x) = \arcsin(x)$

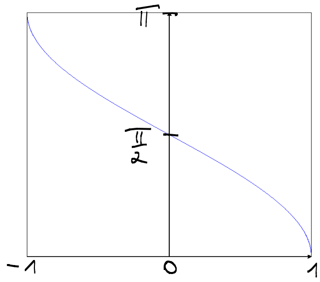




$f_2 : [0, \pi] \to [-1, 1]$ $\quad$ $f_2^{-1} : [-1, 1] \to [0, \pi]$

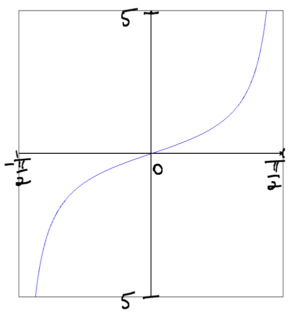
$f_2(x) = \cos(x)$ $\quad$ $f_2^{-1}(x) = \arccos(x)$

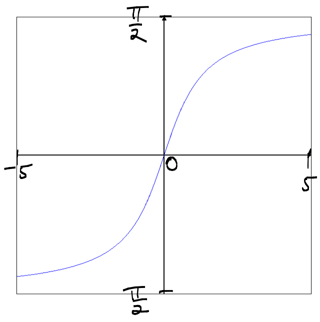




$f_3 : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ $\quad$ $f_3^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$

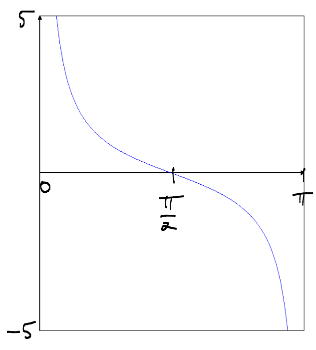
$f_3(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ $\quad$ $f_3^{-1}(x) = \arctan(x)$

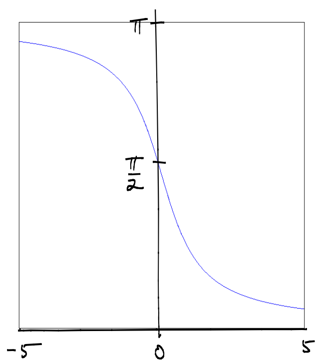




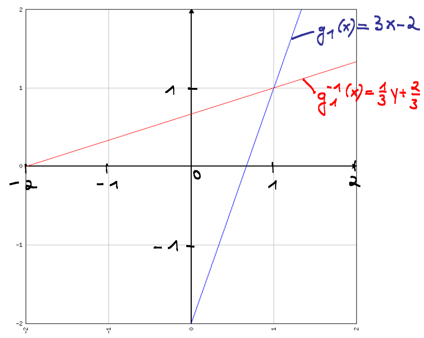
$f_4 : (0, \pi) \to \mathbb{R}$ $\quad$ $f_4^{-1} : \mathbb{R} \to (0, \pi)$

$f_4(x) = \cot(x) = \frac{\cos(x)}{\sin(x)}$ $\quad$ $f_4^{-1}(x) = \operatorname{arccot}(x)$

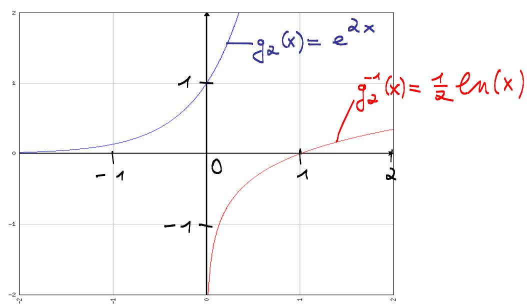




b)
$$
\begin{aligned}
g_1(x) = 3x - 2 &= y\\
\Leftrightarrow \quad 3x &= y + 2\\
\Leftrightarrow \quad x &= \underline{\underline{\tfrac{1}{3}y + \tfrac{2}{3} = g_1^{-1}(y)}}
\end{aligned}
$$



$$
\begin{aligned}
g_2(x) = e^{2x} = y \quad &\Leftrightarrow \quad \ln(e^{2x}) = \ln y\\
&\Leftrightarrow \quad 2x = \ln y\\
&\Leftrightarrow \quad x = \tfrac{1}{2}\ln y = g_2^{-1}(y)
\end{aligned}
$$



$g_3: [3,\infty) \to [-1,\infty)$

$g_3(x) = x^2 - 6x + 8$

$= x - 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 8$

$= (x-3)^2 - 1 \geq -1$

Sei $x \in [3,\infty)$ und

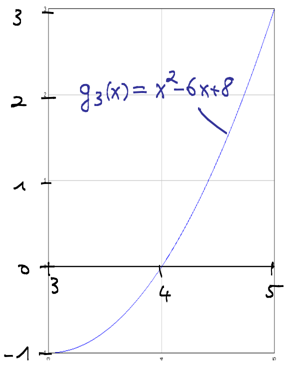
$(x-3)^2 - 1 = y$. (*)

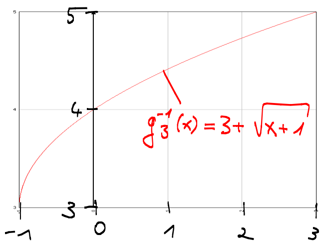
Wegen $x-3 \geq 0$ ist (*) äquivalent zu

$x - 3 = +\sqrt{y+1}$

$\Leftrightarrow \quad x = 3 + \sqrt{y+1} = g_3^{-1}(y)$

$g_3^{-1}: [-1,\infty) \to [3,\infty)$





$g_4: [-\infty,3] \to [-1,\infty)$

$g_4(x) = x^2 - 6x + 8 \overset{(s.o.)}{=} (x-3)^2 - 1 \geq -1$

Sei $x \in [-\infty,3]$ und

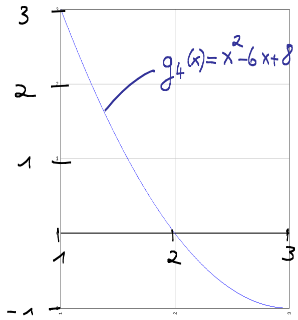
$(x-3)^2 - 1 = y$ (**)

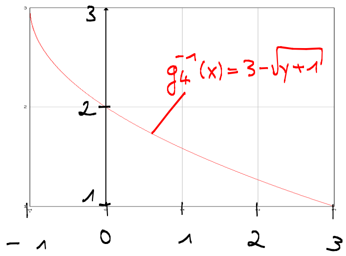
Wegen $x-3 \leq 0$ ist (**) äquivalent zu

$x - 3 = -\sqrt{y+1}$

$\Leftrightarrow \quad x = 3 - \sqrt{y+1} = g_4^{-1}(y)$

$g_4^{-1}: [-1,\infty) \to (-\infty,3]$





Aufgabe 3

$[x]$ rundet zur nächstniedrigeren ganzen Zahl ab.

a) $-\frac{8}{3} = -2{,}666\ldots$, $[-\frac{8}{3}] = -3$

$\sqrt{3} = 1{,}75\ldots$, $[\sqrt{3}] = 1$

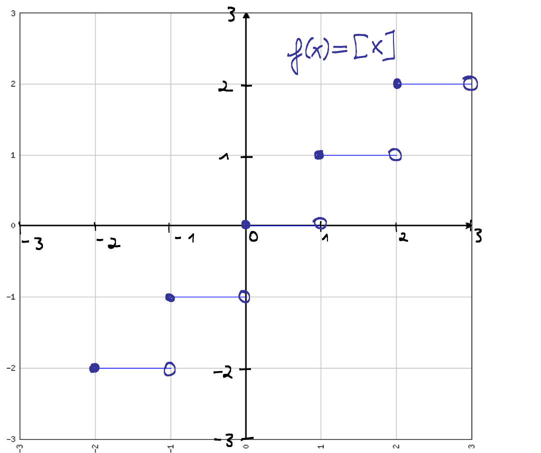
$\pi = 3{,}14\ldots$, $[\pi] = 3$

$\pi^2 = 9{,}86\ldots$, $[\pi^2] = 9$

$e = 2{,}71\ldots$, $[e] = 2$

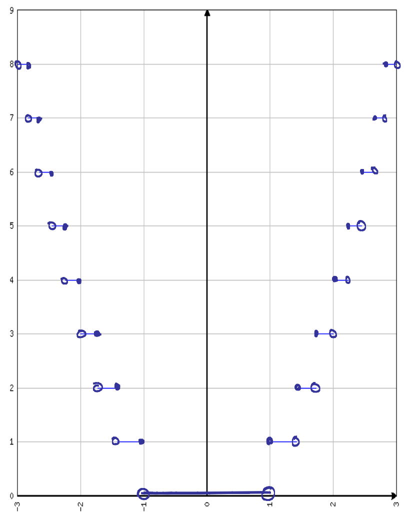
$e^2 = 7{,}38\ldots$ $[e^2] = 7$

b)



Auf allen offenen Intervallen $\ldots, (-1,0), (0,1), (1,2), \ldots$ ist $f(x) = [x]$ stetig.
In allen ganzen Zahlen $0, \pm 1, \pm 2, \ldots$ ist f unstetig.

c)



In allen Punkten $1, \pm\sqrt{2}, \pm\sqrt{3}, \pm\sqrt{4}, \ldots$ ist $g(x) = [x^2]$ unstetig.

D.h. $\mathcal{U} = \{\pm\sqrt{k} \mid k \in \mathbb{N}\}$.

Aber: In $x = 0$ ist g stetig!

Aufgabe 4

$f_1(x) = ax+b$

$$\frac{f_1(x) - f_1(x_0)}{x - x_0} = \frac{ax+b-(ax_0+b)}{x-x_0} = \frac{a\cancel{(x-x_0)}}{\cancel{x-x_0}} = \underline{\underline{a}}$$

$$\Rightarrow f_1'(x_0) = a \quad \forall\, x_0 \in \mathbb{R}.$$

$f_2(x) = x^2$

$$\frac{f_2(x) - f_2(x_0)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} = \frac{\cancel{(x-x_0)}(x+x_0)}{\cancel{x-x_0}} = x + x_0 \to x_0 + x_0 = \underline{\underline{2x_0}} \text{ für } x \to x_0$$

$$\Rightarrow f_2'(x_0) = (x_0^2)' = 2x_0.$$

$$f_2'(1) = 2.$$

$f_3(x) = x^2 - 1$

$$\frac{f_3(x) - f_3(x_0)}{x - x_0} = \frac{x^2 - 1 - (x_0^2 - 1)}{x - x_0} = \frac{x^2 - x_0^2}{x - x_0} \overset{\text{s.o.}}{=} x + x_0 \to \underline{\underline{2x_0}} \text{ für } x \to x_0$$

$$\Rightarrow f_3'(x_0) = (x_0^2 - 1)' = 2x_0$$

$$f_3'(2) = 4$$

$f_4(x) = x^3 + 1$

$$\frac{f_4(x) - f_4(x_0)}{x - x_0} = \frac{x^3 + 1 - (x_0^3 - 1)}{x - x_0} = \frac{x^3 - x_0^3}{x - x_0}$$

Polynomdivision:

$$(x^3 - x_0^3) : (x - x_0) = x^2 + x_0 x + x_0^2$$
$$-(x^3 - x_0 x^2)$$
$$x_0 x^2 - x_0^3$$
$$-(x_0 x^2 - x_0^2 x)$$
$$x_0^2 x - x_0^3$$
$$-(x_0^2 x - x_0^3)$$
$$0$$

Also: $$\frac{f_4(x) - f_4(x_0)}{x - x_0} = x^2 + x_0 x + x_0^2 \to x_0^2 + x_0^2 + x_0^2 = \underline{\underline{3x_0^2}} \text{ für } x \to x_0$$

$$\Rightarrow f_4'(x_0) = (x_0^3 + 1)' = 3x_0^2$$

$$f_4'(1) = 3$$