KLAUSUR-Vorbereitung: Integralrechnung

1. Bestimmen Sie sämtliche Stammfunktionen zu

a) $4x^5-6x^3+8x^2-3x+5$ b) $3\sin x-4\cos x$

c) $2e^x-\frac{5}{x}+1$ d) $\frac{1-2x^2-4x^3}{2x}$

e) $\frac{5}{1+x^2}-\frac{1}{4}x^4$ f) $\frac{-2}{\sqrt{1-x^2}}-\frac{1}{\cos^2 x}$

2.) Lösen Sie die unbestimmten Integrale durch partielle Integration.

$\int(1+2x)\cdot e^{-x}\,dx$, $\int x\cdot\ln x\,dx$, $\int x\cdot\cos x\,dx$

$\int 1\cdot\ln x\,dx$, $\int x\cdot\sin(3x)\,dx$, $\int x\cdot e^x\,dx$

3.) Berechnen Sie folgende Integrale.

$\int_0^1\frac{x}{(1+x^2)^2}\,dx$, $\int\frac{\arctan x}{1+x^2}\,dx$, $\int\frac{8x^3-20x}{x^4-5x^2+4}\,dx$

$\int\frac{(\ln x)^3}{x}\,dx$, $\int\frac{x^2}{\sqrt{1+x^3}}\,dx$, $\int_0^{\pi}\cos^3 x\cdot\sin x\,dx$

4.) Welchen Wert besitzen die folgenden bestimmten Integrale?

$\int_0^1(x^2-4x+3)\,dx$, $\int_1^e\frac{1-x^2}{x}\,dx$, $\int_0^2(x+2)\sin(x^2+4x-6)\,dx$

5.) a) Welchen Flächeninhalt schließt die Parabel $-\frac{1}{4}x^2+4$ mit der x-Achse zwischen den beiden Nullstellen ein?

b) Berechnen Sie die Fläche unter der Kosinuskurve im Intervall $[-\frac{\pi}{2},\frac{\pi}{2}]$.

Lösungen

1.)

a) $F(x) = \frac{2}{3}x^6 - \frac{3}{2}x^4 + \frac{8}{3}x^3 - \frac{3}{2}x^2 + 5x + C$

b) $F(x) = -3\cos x - 4\sin x + C$

c) $F(x) = 2e^x - 5\ln|x| + x + C$

d) $f(x) = \frac{1}{2x} - x - 2x^2$

$F(x) = \frac{1}{2}\ln|x| - \frac{1}{2}x^2 - \frac{2}{3}x^3 + C$

e) $F(x) = 5\arctan x - \frac{1}{20}x^5 + C$

f) $F(x) = -2\arcsin x - \tan x + C$

2.)

$$\int \underbrace{(1+2x)}_{u}\,\underbrace{e^{-x}}_{v'}\,dx = (1+2x)(-e^{-x}) - 2\int(-e^{-x})\,dx$$

$$= -e^{-x}(1+2x) - 2e^{-x} + C$$

$$= -e^{-x}(3+2x) + C$$

$$\int \underbrace{x}_{u'}\cdot\underbrace{\ln x}_{v}\,dx = \underbrace{\tfrac{1}{2}x^2}_{u}\cdot\underbrace{\ln x}_{v} - \int \overbrace{\underbrace{\tfrac{1}{2}x^2}_{u}\cdot\underbrace{\tfrac{1}{x}}_{v'}}^{x}\,dx$$

$$= \tfrac{1}{2}x^2\cdot\ln x - \tfrac{1}{2}\cdot\tfrac{1}{2}x^2 + C$$

$$= \tfrac{1}{2}x^2\left(\ln x - \tfrac{1}{2}\right) + C$$

$$\int \underset{u}{x} \cdot \underset{v'}{\cos x}\, dx = x \sin x - \int \sin x$$
$$= x \sin x + \cos x + C$$

$$\int \underset{u'}{1} \cdot \underset{v}{\ln x}\, dx = x \ln x - \int \overbrace{x \cdot \frac{1}{x}}^{1} dx$$
$$= x \ln x - x + C$$

$$\int \underset{u}{x} \cdot \underset{v'}{\sin(3x)}\, dx = x \cdot \frac{1}{3}\left(-\cos(3x)\right) - \int \left(-\frac{1}{3}\cos(3x)\right) dx$$
$$= -\frac{x}{3}\cos(3x) + \frac{1}{3}\int \cos(3x)\, dx$$
$$= -\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + C$$

$$\int \underset{u}{x}\, \underset{v'}{e^x}\, dx = x e^x - \int e^x dx$$
$$= x e^x - e^x + C$$
$$= (x-1)e^x + C$$

3.)

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}\int \frac{1}{(\underbrace{1+x^2}_{t})^2}\cdot \underbrace{2x}_{t'}\,dx$$

Substitution $t = 1+x^2$

$$= \frac{1}{2}\int \frac{1}{t^2}\,dt$$

$$= \frac{1}{2}\int t^{-2}\,dt$$

$$= \frac{1}{2}\,\frac{1}{-1}\,t^{-1} + C$$

$$= -\frac{1}{2t} + C$$

$$= -\frac{1}{2(1+x^2)} + C$$

$$\int_0^1 \frac{x}{(1+x^2)^2}\,dx = \left[-\frac{1}{2(1+x^2)}\right]_0^1 = -\frac{1}{4} - \left(-\frac{1}{2}\right) = \underline{\underline{\frac{1}{4}}}$$

$$\int \underbrace{\arctan x}_{t}\cdot \underbrace{\frac{1}{1+x^2}}_{t'}\,dx = \int t\,dt$$

Subst. $t = \arctan x$

$$= \frac{1}{2}t^2 + C$$

$$= \underline{\underline{\frac{1}{2}(\arctan x)^2 + C}}$$

$$\int \frac{8x^3-20x}{x^4-5x^2+4}\,dx = 2\cdot\int \underbrace{\frac{1}{x^4-5x^2+4}}_{t} \cdot \underbrace{(4x^3-10x)}_{t'}\,dx$$

Substitution $t = x^4-5x^2+4$

$$= 2\cdot\int \frac{1}{t}\,dt$$

$$= 2\cdot \ln|t| + C$$

$$= \underline{\underline{2\cdot \ln|x^4-5x^2+4| + C}}$$

$$\int \frac{(\ln x)^3}{x}\,dx = \int (\underbrace{\ln x}_{t})^3 \cdot \underbrace{\frac{1}{x}}_{t'}\,dx$$

Substitution $t = \ln x$

$$= \int t^3\,dt$$

$$= \frac{1}{4}t^4 + C$$

$$= \underline{\underline{\frac{1}{4}(\ln x)^4 + C}}$$

$$\int \frac{x^2}{\sqrt{1+x^3}} = \frac{1}{3} \int \frac{1}{\sqrt{\underbrace{1+x^3}_{t}}} \cdot \underbrace{3x^2}_{t'} \, dx$$

Substitution
$t = 1+x^3$

$$= \frac{1}{3} \int \frac{1}{\sqrt{t}} dt = \frac{1}{3} \int t^{-\frac{1}{2}} dt$$

$$= \frac{1}{3} \frac{1}{-\frac{1}{2}+1} t^{-\frac{1}{2}+1} + C$$

$$= \frac{2}{3} t^{\frac{1}{2}} + C = 2\sqrt{t} + C$$

Rücksubst.
$t = 1-x^3$ $\rightarrow$ $$= \underline{\underline{\frac{2}{3} \sqrt{1+x^3} + C}}$$

$$\int \cos^3 x \cdot \sin x \, dx = - \int (\underbrace{\cos x}_{t})^3 \cdot \underbrace{(-\sin x)}_{t'} dx$$

Subst. $t = \cos x$

$$= - \int t^3 dt$$

$$= -\frac{1}{4} t^4 + C$$

$$= -\frac{1}{4} \cos^4 x$$

$$\int_0^{\pi} \cos^3 x \cdot \sin dx = -\frac{1}{4}\cos^4 \pi - \left(-\frac{1}{4}\cos^4 0\right)$$

$$= -\frac{1}{4}(-1)^4 + \frac{1}{4}1^4 = 0$$

4.) $$\int_0^1 (x^2 - 4x + 3)dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_0^1$$

$$= \left(\frac{1}{3} - 2 + 3\right) - 0$$

$$= \underline{\underline{\frac{4}{3}}}$$

$$\int_1^e \frac{1-x^2}{x}dx = \int_1^e \left(\frac{1}{x} - x\right)dx = \left[\ln|x| - \frac{1}{2}x^2\right]_1^e$$

$$= \underbrace{\ln e}_{1} - \frac{1}{2}e^2 - \left(\underbrace{\ln 1}_{0} - \frac{1}{2}\cdot 1\right)$$

$$= 1 - \frac{1}{2}e^2 + \frac{1}{2}$$

$$= \underline{\underline{\frac{3}{2} - \frac{1}{2}e^2}}$$

$$\int (x+2)\sin(x^2 + 4x - 6)dx$$

$$= \frac{1}{2}\int \underbrace{(2x+4)}_{t'}(\underbrace{x^2 + 4x - 6}_{t})dx$$

Substitution $t = x^2 + 4x - 6$

$$= \frac{1}{2} \int \sin t \, dt$$

$$= \frac{1}{2} (-\cos t) + C$$

$$= -\frac{1}{2} \cos(x^2 + 4x - 6) + C$$

$$\int_0^2 (x+2) \sin(x^2+4x-6) \, dx = \left[-\frac{1}{2} \cos(x^2+4x-6) \right]_0^2$$

$$= -\frac{1}{2} \cos 6 - \left(-\frac{1}{2} \underbrace{\cos(-6)}_{\cos 6} \right)$$

$$= -\frac{1}{2} \cos 6 + \frac{1}{2} \cos 6$$

$$= \underline{\underline{0}}$$

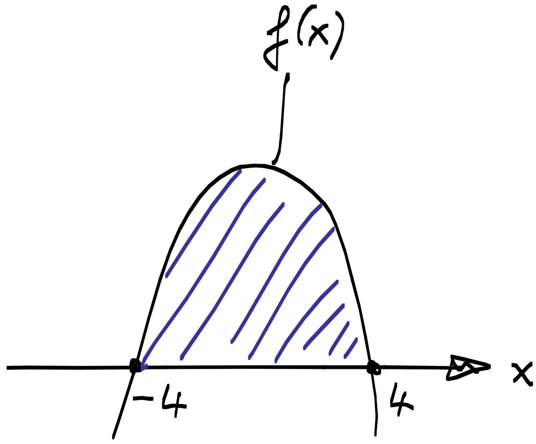
5.)

a) $f(x) = -\frac{1}{4}x^2 + 4 = 0$

$\Leftrightarrow \quad \frac{1}{4}x^2 = 4$

$\Leftrightarrow \quad x^2 = 16$

$\Leftrightarrow \quad x = -4 \vee x = 4$



$$\int_{-4}^{4} -\frac{1}{4}x^2 + 4\,dx = \left[-\frac{1}{12}x^3 + 4x\right]_{-4}^{4}$$

$$= -\frac{1}{\cancel{12}_{3}} \cdot 4^{\cancel{3}\,2} + 16 - \left(+\frac{1}{\cancel{12}_{3}} 4^{\cancel{3}\,2} - 16\right)$$

$$= 32 - \frac{2}{3} \cdot 16 = 21{,}\overline{3}$$

b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 1 - (-1) = 2$