KLAUSUR-Vorbereitung: Reihen und Produkte

1.) Berechnen Sie

(a) $\sum_{k=1}^{5} \frac{k-1}{k+2}$ (b) $\sum_{k=1}^{4} (2k-5)^2$ (c) $\sum_{k=1}^{5} (2k+3-\frac{4}{k})$

2.) Vereinfachen und berechnen Sie.

(a) $\sum_{k=1}^{100} (15+k)$ (b) $\sum_{k=1}^{80} (2k+1)$

(c) $\sum_{k=1}^{90} (5+4(k-1))$ (d) $\sum_{k=1}^{100} (1+k)^2 - \sum_{k=1}^{100} (1-k)^2$

(e) $\sum_{k=1}^{50} (k^2+2k-3) + \sum_{k=1}^{50} (3k^2+5k+8) - \sum_{k=1}^{50} (4k^2+6k-10)$

3.) Beweisen Sie die Identität

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad n \in \mathbb{N}.$$

Tipp: Formen Sie $2 \cdot \sum_{k=1}^{n} k$ um.

4.) Berechnen Sie

(a) $1+3+3^2+3^3+\ldots+3^{10}$

(b) $1+2+2^2+2^3+\ldots+2^n$

(c) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{1024}$

(d) $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$

(e) $\sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k$

5.) Beweisen Sie die Identität

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \quad \text{für } q \in \mathbb{R} \setminus \{1\},\ n \in \mathbb{N}_0 .$$

Tipp: Formen Sie $(1-q) \sum_{k=0}^{n} q^n$ um.

6.) Berechnen Sie

(a) $\prod_{k=10}^{13} k$ (b) $\prod_{k=1}^{6} \frac{1}{k}$ (c) $\prod_{k=1}^{15} \frac{k-1}{k^2+1}$

(d) $\prod_{k=20}^{30} \frac{(k+1)(k-1)}{k^2-1}$ (e) $\prod_{k=1}^{20} \frac{k}{k+2}$

7.) Berechnen Sie

(a) $\binom{12}{5}$, $\binom{12}{7}$, $\binom{20}{4}$

(b) die ersten 4 Summanden von $(a+b)^8$

(c) $(m+n)^3 - (m-n)^3$

(d) $(5a - 3x^2)^3$

Lösungen

1.)

(a) $\sum_{k=1}^{5} \frac{k-1}{k+2} = 0 + \frac{1}{4} + \frac{2}{5} + \frac{3}{6} + \frac{4}{7} = \frac{241}{140}$

(b) $\sum_{k=1}^{4} (2k-5)^2 = (-3)^2 + (-1)^2 + 1^2 + 3^2 = 20$

(c) $\sum_{k=1}^{5} \left(2k + 3 - \frac{4}{k}\right) = 2 \sum_{k=1}^{5} k + 5 \cdot 3 - 4 \sum_{k=1}^{5} \frac{1}{k}$

$= 2(1+2+3+4+5) + 15 - 4\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)$

$= 2 \cdot 15 + 15 - 4 \cdot \frac{137}{60}$

$= 45 - \frac{137}{15}$

$= \frac{538}{15}$

$= 35 \frac{13}{15}$

2.)

(a) $$\sum_{k=1}^{100}(15+k) = \sum_{k=1}^{100}15 + \sum_{k=1}^{100}k$$

$$= 15\cdot 100 + \frac{100\cdot 101}{2}$$

$$= 1500 + 5050$$

$$= \underline{\underline{6550}}$$

(b) $$\sum_{k=1}^{80}(2k+1) = 2\sum_{k=1}^{80}k + \sum_{k=1}^{80}1$$

$$= \cancel{2}\cdot\frac{80\cdot 81}{\cancel{2}} + 80$$

$$= 6480 + 80$$

$$= \underline{\underline{6560}}$$

(c) $$\sum_{k=1}^{90}(5+4(k-1)) = 5\cdot\sum_{k=1}^{90}1 + 4\cdot\sum_{k=1}^{90}(k-1)$$

$$= 450 + 4\left(\sum_{k=1}^{90}k - \sum_{k=1}^{90}1\right)$$

$$= 450 + 4\left(\underbrace{\frac{90\cdot 91}{2}}_{4095} - 90\right)$$

$$= 450 + 16020$$

$$= \underline{\underline{16470}}$$

(d) $$\sum_{k=1}^{100}\left(\cancel{1}+2k+\cancel{k^2}-(\cancel{1}-2k+\cancel{k^2})\right)$$

$$=\sum_{k=1}^{100}4k$$

$$=4\sum_{k=1}^{100}k$$

$$=\overset{2}{\cancel{4}}\cdot\frac{100\cdot101}{\cancel{2}}$$

$$=2\cdot 10100$$

$$=\underline{\underline{20200}}$$

(e) $$\sum_{k=1}^{50}\left(\cancel{k^2}+2k-3+\cancel{3k^2}+5k+8-\cancel{4k^2}-6k+10\right)$$

$$=\sum_{k=1}^{50}(k+15)=\sum_{k=1}^{50}k+\sum_{k=1}^{50}15$$

$$=\sum_{k=1}^{50}k+15\cdot\sum_{k=1}^{50}1$$

$$=\frac{50\cdot51}{2}+15\cdot50$$

$$=1275+750$$

$$=\underline{\underline{2025}}$$

3.)

Beweis:

$$2 \cdot \sum_{k=1}^{n} k = \begin{aligned}[t] &1 + 2 + \dots + (n-1) + n \\ + \; &n + (n-1) + \dots + 2 + 1 \end{aligned}$$

$$= (n+1) + (n+1) + \dots + (n+1) + (n+1)$$

$$= n \cdot (n+1)$$

$$\Rightarrow \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}. \qquad \square$$

4.)

(a) $1+3+3^2+\ldots+3^{10}=\sum_{k=0}^{10} 3^k = \frac{1-3^{11}}{1-3}$

$= \frac{3^{11}-1}{3-1} = \frac{1}{2}(3^{11}-1) = \underline{\underline{88\,573}}$

(b) $1+2+2^2+\ldots+2^n = \sum_{k=0}^{n} 2^k = \frac{1-2^{n+1}}{1-2}$

$= \frac{2^{n+1}-1}{2-1} = \underline{\underline{2^{n+1}-1}}$

(c) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{1024} = \sum_{k=0}^{10}\left(\frac{1}{2}\right)^k$

$= \frac{1-\left(\frac{1}{2}\right)^{11}}{1-\frac{1}{2}} = \frac{1-\frac{1}{2048}}{\frac{1}{2}}$

$= \underline{\underline{2-\frac{1}{1024}}}$

(d) $\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = \frac{1}{1-\frac{2}{3}} = 3$

(e) $\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$

5.)

Beweis:

$$(1-q)\sum_{k=0}^{n} q^k = \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1}$$

$$= 1 + \cancel{q} + \cancel{q^2} + \dots + \cancel{q^n}$$

$$- \left(\cancel{q} + \cancel{q^2} + \dots + \cancel{q^n} + q^{n+1} \right)$$

$$= 1 - q^{n+1}$$

$$\Rightarrow \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}. \quad \square$$

6.)

(a) $\prod_{k=10}^{13} k = 10 \cdot 11 \cdot 12 \cdot 13 = 17160$

(b) $\prod_{k=1}^{6} \frac{1}{k} = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{1}{6} = \frac{1}{720}$

(c) $\prod_{k=1}^{15} \frac{k-1}{k^2+1} = 0$ mit dem ersten Faktor verschwindet das Produkt

(d) $\prod_{k=20}^{30} \frac{(k-1)(k-1)}{k^2-1} = \prod_{k=20}^{30} 1 = 1$

e) $\prod_{k=1}^{20} \frac{k}{k+2} = \frac{1}{\cancel{3}} \cdot \frac{2}{\cancel{4}} \cdot \frac{\cancel{3}}{\cancel{5}} \cdot \frac{\cancel{4}}{\cancel{6}} \cdots \frac{\cancel{18}}{\cancel{20}} \cdot \frac{\cancel{19}}{21} \cdot \frac{\cancel{20}}{22}$

$$= \frac{2}{21 \cdot 22} = \frac{1}{231}$$

7.)

(a) $\binom{12}{5} = \frac{\overset{6}{\cancel{12}} \cdot 11 \cdot \overset{2}{\cancel{10}} \cdot \overset{3}{\cancel{9}} \cdot \overset{2}{\cancel{8}}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}} = 72 \cdot 11 = 792 = \binom{12}{7}$

$\binom{20}{4} = \frac{\overset{5}{\cancel{20}} \cdot 19 \cdot \overset{\cancel{6}\,3}{\cancel{18}} \cdot 17}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4}} = 5 \cdot 19 \cdot 3 \cdot 17 = 4845$

(b) $(a+b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + \ldots$

(c) $(m+n)^3 - (m-n)^3$

$= \cancel{m^3} + 3m^2n + \cancel{3mn^2} + n^3$

$-(\cancel{m^3} - 3m^2n + \cancel{3mn^2} - n^3)$

$= 6m^2n + 2n^3$

(d) $(5a-3x^2)^3 = (5a)^3 - 3\cdot(5a)^2(3x^2) + 3(5a)(3x^2)^2 - (3x^2)^3$

$= 125a^3 - 225a^2x^2 + 135ax^4 - 27x^6$