KLAUSUR-Vorbereitung: Komplexe Zahlen

1.) Man berechne $z+w$, $z\cdot w$, $\frac{z}{w}$ und $|z|, |w|$ jeweils für
(a) $z = 4-5i$, $w = 4+5i$,
(b) $z = i$, $w = -2-4i$.

2.) Berechnen Sie $\operatorname{Re}(z)$, $\operatorname{Im}(z)$ und $|z|$ jeweils für

$$z = \frac{1}{1+i}, \quad z = \left(\frac{1+i}{1-i}\right)^2, \quad z = (1+i)^4.$$

3.) Vereinfachen Sie:

$$z = \frac{1+\frac{1}{i}}{1-\frac{1}{i}}, \quad z = \frac{\frac{5}{2}+\frac{i}{2}}{2+\frac{1}{1-i}}, \quad z = \frac{(1+2i)(2-i)+1}{(2-i)^2-2+i}$$

$$z = \frac{(6-2i)(1+i)}{(2+i)(2+2i)}, \quad z = \frac{1}{i^5}+\frac{1}{i^7}$$

4.) Skizzieren Sie in der komplexen Zahlenebene die Punktmengen, die den folgenden Ungleichungen entsprechen:

(a) $|z| \leq 2$ (b) $|z| > 2$

(c) $|z-z_0| \leq 1$ (d) $1 \leq |z-2| \leq 2$

(e) $\operatorname{Re}(z) \geq 1$ (f) $\operatorname{Im}(z) > 1$

Lösung:

1.) (a) $z+w = 8$

$z \cdot w = 16 + 25 = 41$

$$\frac{z}{w} = \frac{(4-5i)(4-5i)}{(4+5i)(4-5i)} = \frac{16-40i-25}{16+25}$$

$$= \frac{-9-40i}{41} = -\frac{9}{41} - \frac{40}{41}i$$

$$\frac{z}{w} = \frac{z}{\bar{z}} = \frac{z \cdot z}{|z|^2} \quad \text{da } w = \bar{z} \text{ ist}$$

$$|z| = \sqrt{41} = |w|$$

(b) $z+w = -2-3i$

$z \cdot w = -2i - 4i = 4-2i$

$$\frac{z}{w} = \frac{i(-2+4i)}{(-2-4i)(-2+4i)} = \frac{-4-2i}{4+16} = -\frac{1}{5} - \frac{1}{10}i$$

$$|z| = \sqrt{1^2+0^2} = 1, \quad w = \sqrt{4+16} = \sqrt{20}$$

2.) $$z = \frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i$$

$$\Rightarrow Re(z) = \frac{1}{2}, \quad Im(z) = -\frac{1}{2}, \quad |z| = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$$z = \left(\frac{1+i}{1-i}\right)^2 = \left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^2 = \left(\frac{1+2i-1}{1+1}\right)^2 = i^2 = -1$$

$$\Rightarrow Re(z) = -1, \quad Im(z) = 0, \quad |z| = 1$$

$$z = (1+i)^4 = (1+2i-1)^2 = (2i)^2 = -4$$

$$\Rightarrow Re(z) = -4, \quad Im(z) = 0, \quad |z| = \sqrt{(-4)^2} = 4$$

3.) $\frac{1+\frac{1}{i}}{1-\frac{1}{i}} = \frac{i+1}{i-1} = \frac{(i+1)(-i-1)}{(i-1)(-i-1)} = \frac{-(i+1)^2}{1+1}$

$$= -\frac{-1+2i+1}{2} = -i$$

$$\frac{\frac{5}{2}+\frac{i}{2}}{2+\frac{1}{1-i}} = \frac{1}{2}\,\frac{(5+i)(1-i)}{2(1-i)+1} = \frac{1}{2}\,\frac{5-5i+i+1}{3-2i}$$

$$= \frac{1}{2}\,\frac{6-4i}{3-2i}\cdot\frac{3+2i}{3+2i}$$

$$= \frac{1}{2}\,\frac{18+\cancel{12i}-\cancel{12i}+8}{9+4}$$

$$= \frac{1}{2}\,\frac{26}{13} = 1$$

$$\frac{(1+2i)(2-i)+1}{(2-i)^2-2+i} = \frac{2-i+4i+2+1}{4-4i-1-2+i} = \frac{5+3i}{1-3i}$$

$$= \frac{(5+3i)(1+3i)}{(1-3i)(1+3i)} = \frac{5+15i+3i-9}{1+9}$$

$$= \frac{-4+18i}{10} = -\frac{2}{5}+\frac{9}{5}i$$

$$\frac{(6-2i)(1+i)}{(2+i)(2+2i)} = \frac{6+6i-2i+2}{4+4i+2i-2} = \frac{8+4i}{2+6i}$$

$$= \frac{4+2i}{1+3i}\,\frac{1-3i}{1-3i} = \frac{4-12i+2i+6}{1+9} = \frac{10-10i}{10}$$

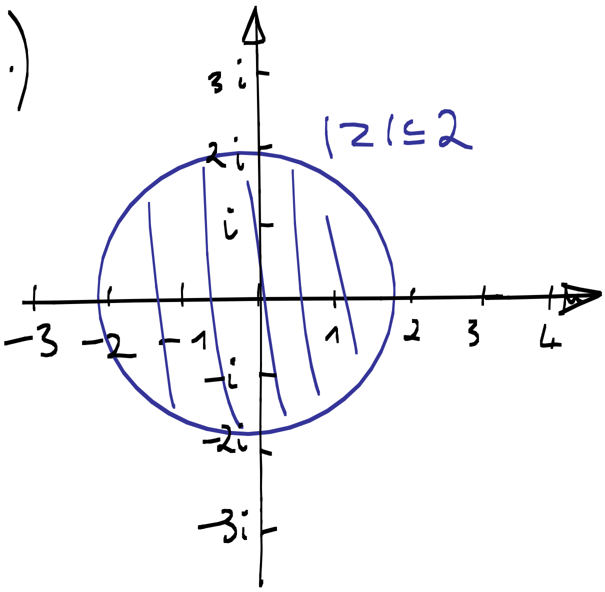
$$= 1-i$$

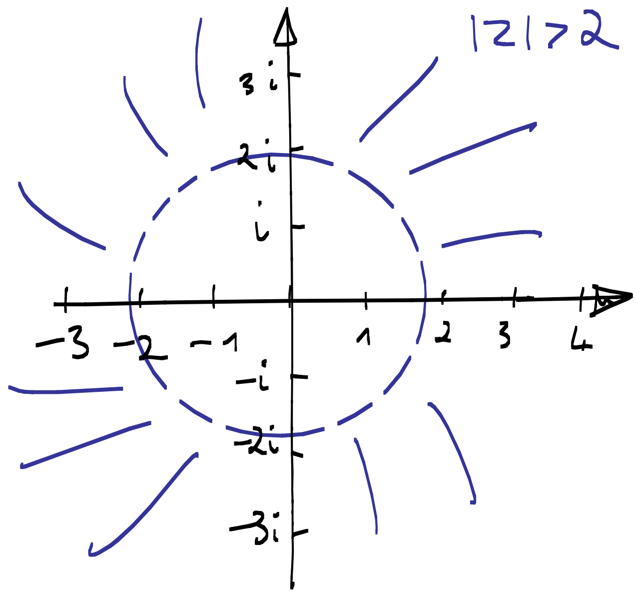
$$z = \frac{1}{i^5} + \frac{1}{i^7} = \frac{1}{i \cdot i^4} + \frac{i}{i^4 \cdot i^4} = \frac{1}{i} + i = -i + i = 0$$

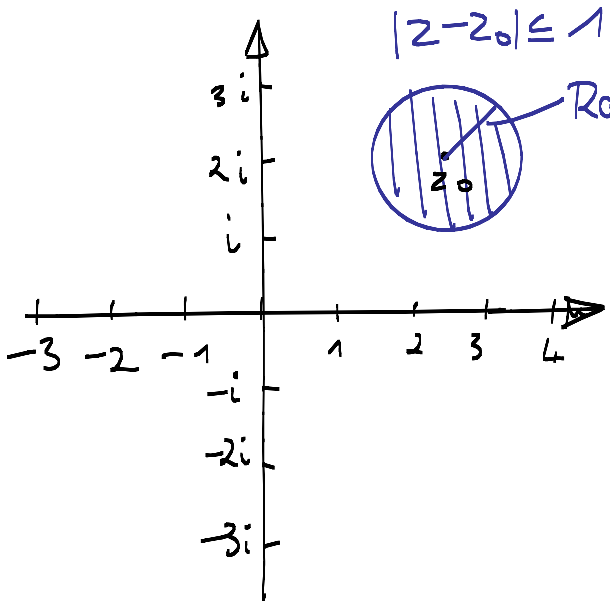
Nebenrechnung: $i^4 = (i^2)^2 = (-1)^2 = 1$

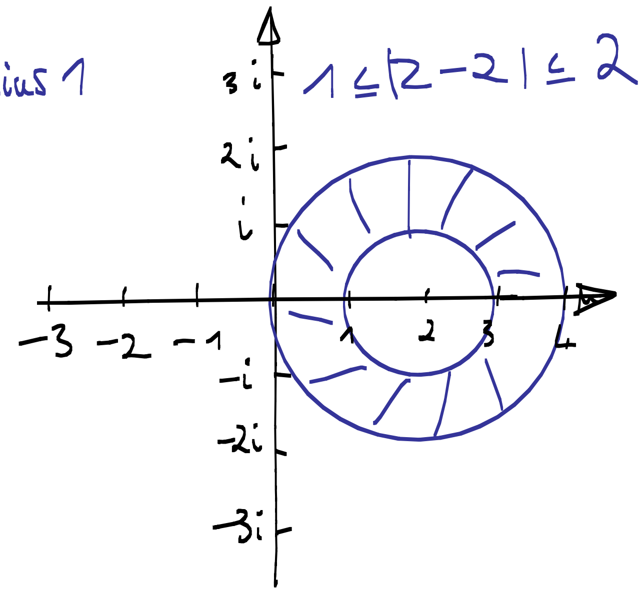
$$\frac{1}{i} = \frac{i}{i \cdot i} = \frac{i}{-1} = -i$$

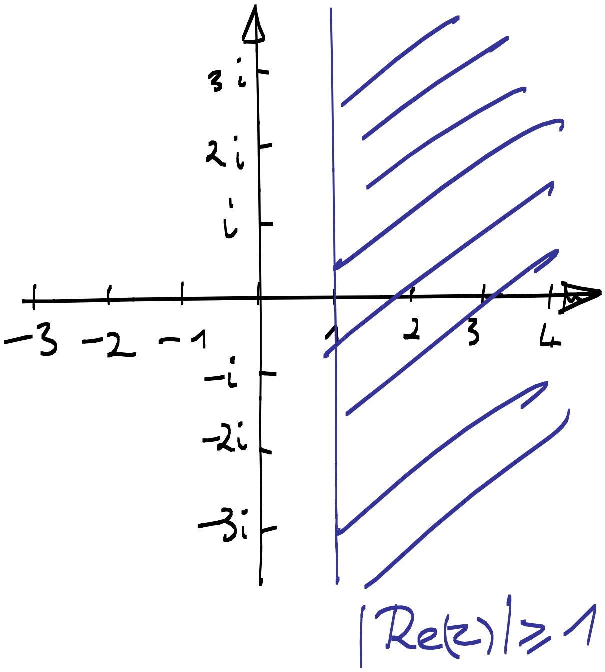
4.)

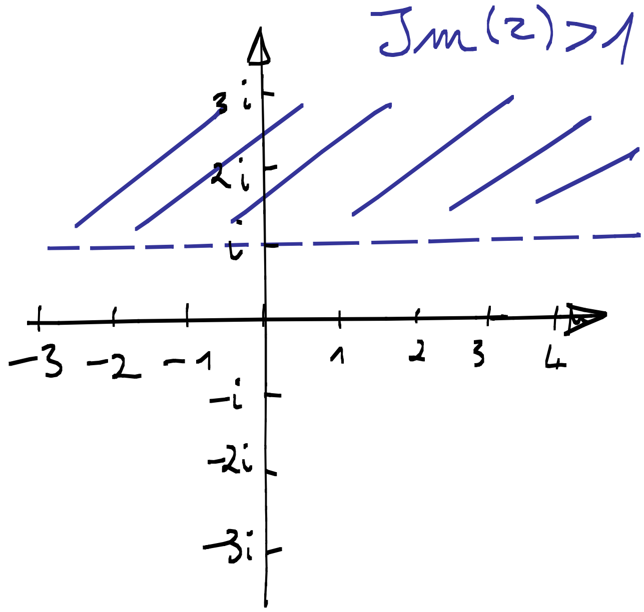









3i
2i
i
−3 −2 −1
1 2 3 4
−i
−2i
−3i
|Re(z)|≥1


Jm(z)>1
3i
2i
i
−3 −2 −1
1 2 3 4
−i
−2i
−3i